

Comparison of Two Numerical Methods for Computation of American Type of the Floating Strike Asian Option

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Abstract. We present a numerical approach for solving the free boundary problem for the Black-Scholes equation for pricing American style of floating strike Asian options. A fixed domain transformation of the free boundary problem into a parabolic equation defined on a fixed spatial domain is performed. As a result a nonlinear time-dependent term is involved in the resulting equation. Two new numerical algorithms are proposed. In the first algorithm a predictor-corrector scheme is used. The second one is based on the Newton method. Computational experiments, confirming the accuracy of the algorithms are presented and discussed.

1 Introduction

In this paper we consider the problem of pricing American style Asian options, analyzed by Bokes and the second author in [1] (see also [11]). Asian options belong to the group of the so-called path-dependent options. Their pay-off diagrams depend on the spot value of the underlying asset during the whole or some part(s) of the life span of the option. Among path-dependent options, Asian option depend is on the arithmetic or geometric average of spot prices of the underlying asset. During the last decade, the problem of solving the American option problem numerically has been subject for intensive research [1,6,9,10,13] (see also [11] for overview). A comprehensive introduction to this topic can be found in [6]. Comparison of various analytical and numerical approximation methods of calculation of the early exercise boundary a position of the American put option paying zero dividends is given in [7]. An improvement of Han and Wu's algorithm [4] is described in [14]. Our goal is to propose and investigate two front-fixing numerical algorithms for solving free boundary value problems. The front-fixing method has been successfully applied to a wide range of applied problems arising from physics and engineering, cf. [3,8] and references therein. The basic idea is to remove the moving boundary by a transformation of the involved variables. Transformation techniques were used in the analysis and numerical computation of the early exercise boundary in the context of American style of vanilla options [10] as well as Asian floating strike options [1,11,12]. In comparison to

the existing computational method [1] we do not replace the algebraic constraint by its equivalent integral form (see [1,12] for details) which is computationally more involved. In this paper we solve the corresponding parabolic equation with an algebraic constraint directly as it was proposed in [11]. The approach presented in [11] however suffered from the necessity of taking very small time discretization steps. Here we overcome this difficulty by proposing two new numerical approximation algorithms (see Section 4). They are based on the novel technique proposed by the first author and Valkov in [5]. We extend this approach for American style of Asian options. In Section 5, a numerical example illustrating the capability of our algorithms are discussed.

2 The Free Boundary Problem

Following the classical Black-Scholes theory, the second author and Bokes [1] analyzed the problem of pricing Asian options with arithmetically averaged strike price by means of a solution to a parabolic PDE with a free boundary $S_f(t, A)$:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} + \frac{S - A}{t} \frac{\partial V}{\partial A} - rV = 0, \quad (1)$$

$0 < t < T$, $0 < S < S_f(t, A)$, satisfying the boundary conditions

$$V(t, 0, A) = 0, \quad \text{for any } A > 0 \text{ and } 0 < t < T, \quad (2)$$

$$\frac{\partial V}{\partial S}(t, S_f(t, A), A) = 1, \quad V(t, S_f(t, A), A) = S_f(t, A) - A, \quad (3)$$

and the terminal condition (terminal pay-off condition) at the maturity time T :

$$V(T, S, A) = \max(S - A, 0), \quad S, A > 0. \quad (4)$$

Here $S > 0$ is the stock price, $A > 0$ is the averaged strike price, $r > 0$ is the risk-free interest rate, $q > 0$ is a continuous dividend rate and $\sigma > 0$ is the volatility of the underlying asset returns. The arithmetically averaged price $A = A_t$ calculated from the price path $\{S_u, u \in [0, T]\}$ at the time t is defined as $A_t = \frac{1}{t} \int_0^t S_u du$. For floating strike Asian options, it is well known (see e.g. [6,2,1]) that one can perform a dimension reduction by introducing a new time variable $\tau = T - t$ and a similarity variable x defined as: $x = A/S$, $W(x, \tau) = V(t, S, A)/A$. The spatial domain for the reduced equation is given by $1/\rho(\tau) < x < \infty$, $\tau \in (0, T)$, $\rho(\tau) = S_f(T - \tau, A)/A$. Following ([10,13,1]), we can apply the Landau fixed domain transformation for the free boundary problem by introducing a new state variable ξ and an auxiliary function $\Pi(\xi, \tau) = W(x, \tau) + x \frac{\partial W}{\partial x}(x, \tau)$, representing a synthetic portfolio. Here $\xi = \ln(\rho(\tau)x)$. In [1,10,13] it is shown that under suitable regularity assumptions on the input data the free boundary problem (1)–(4) can be transformed

into the initial boundary value problem for parabolic PDE:

$$\frac{\partial \Pi}{\partial \tau} + \alpha(\xi, \tau) \frac{\partial \Pi}{\partial \xi} - \frac{\sigma^2}{2} \frac{\partial^2 \Pi}{\partial \xi^2} + \beta(\xi, \tau) \Pi = 0, \quad \xi > 0, \quad \tau \in (0, T), \quad (5)$$

$$\Pi(0, \tau) = -1, \Pi(\infty, \tau) = 0, \quad \Pi(\xi, 0) = \begin{cases} -1, & \text{for } \xi < \ln \rho(0), \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

The coefficients α and β are defined as follows:

$$\alpha(\xi, \tau) = \frac{\dot{\rho}(\tau)}{\rho(\tau)} + r - q - \frac{\sigma^2}{2} - \frac{\rho(\tau)e^{-\xi} - 1}{T - \tau}, \quad \beta(\xi, \tau) = r + \frac{1}{T - \tau}. \quad (7)$$

According to [1] the free boundary function $\rho(\tau)$ and the solution Π should fulfill the constraint:

$$\rho(\tau) = \frac{1 + r(T - \tau) + \frac{\sigma^2}{2}(T - \tau) \frac{\partial \Pi}{\partial \xi}(0, \tau)}{1 + q(T - \tau)}, \quad \rho(0) = \max \left(\frac{1 + rT}{1 + qT}, 1 \right). \quad (8)$$

As for derivation of the initial free boundary position $\rho(0)$ in (8) we refer to [1] or [6,2]. A solution Π to the problem (5)-(8) is continuous for $t > 0$. The discontinuity appears only at the point $P^* = (\ln(\rho(0)), 0)$. The derivatives of the solution exist and are sufficiently smooth in $[0, L] \times [0, T)$, outside of a small neighbourhood of P^* . Another important fact to emphasize is that for times $t \rightarrow 0^+$ (i.e. when $\tau \rightarrow T$) the coefficients α, β become unbounded.

3 Finite Difference Schemes

In order to solve the problem (5)-(8) numerically, we introduce L which is sufficiently large upper limit of values of the ξ variable (a safe choice is to take L is equal to five times $\ln(\rho(0))$), where we prescribe $\Pi(L, \tau) = 0$. Next, for given positive integers N and M we define the uniform meshes: $\bar{\omega}_h = \{0\} \cup \{L\} \cup \omega_h$, $\omega_h = \{\xi_i = ih, i = 1, \dots, (N-1), h = L/N\}$ and $\bar{\omega}_k = \{0\} \cup \{T\} \cup \omega_k$, $\omega_k = \{\tau_j = jk, j = 1, \dots, (M-1), k = T/M\}$. Our goal is to define a finite difference method which is suitable for computing $y_i^j \approx \Pi(\xi_i, \tau_j)$ for $(\xi_i, \tau_j) \in \omega_h \times \omega_k$ and associated front position $z^j \approx \rho(\tau_j)$ for $\tau_j \in \omega_k$. The implicit difference scheme has the following form:

$$\frac{y_i^{j+1} - y_i^j}{k} + \alpha_i^{j+1} \frac{y_{i+1}^{j+1} - y_{i-1}^{j+1}}{2h} - \frac{\sigma^2}{2} \frac{y_{i+1}^{j+1} - 2y_i^{j+1} + y_{i-1}^{j+1}}{h^2} + \beta^{j+1} y_i^{j+1} = 0, \quad (9)$$

$$y_0^{j+1} = -1, \quad y_N^{j+1} = 0; \quad y_i^0 = \begin{cases} -1, & \text{for } \xi_i \leq \ln(\rho(0)), \\ 0, & \text{otherwise;} \end{cases} \quad (10)$$

$$\alpha_i^{j+1} = \frac{z^{j+1} - z^j}{k z^{j+1}} + r - q - \frac{\sigma^2}{2} - \frac{z^{j+1} \exp(-\xi_i) - 1}{T - \tau_{j+1}}, \quad \beta^{j+1} = r + \frac{1}{T - \tau_{j+1}}, \quad (11)$$

$$z^{j+1} - \frac{1 + r(T - \tau_{j+1})}{1 + q(T - \tau_{j+1})} - \frac{\sigma^2}{2} \frac{T - \tau_{j+1}}{1 + q(T - \tau_{j+1})} \frac{-3y_0^{j+1} + 4y_1^{j+1} - y_2^{j+1}}{2h} = 0. \quad (12)$$

For the initial condition for the free boundary we have $z^0 = \rho(0)$. An algebraic nonlinear system of equations can be derived from (9) for $i = 1, \dots, N-1$, (10) and (12). In [9] the authors apply implicit finite difference scheme, semi-implicit scheme and upwind explicit scheme for the American put option, combining with the penalty method. The time step parameter for the explicit case is very small, $k = 5.0 \cdot 10^{-6}$. Therefore in this work we consider the case of a fully implicit scheme. One can also apply a scheme of the Crank-Nicolson type.

4 Numerical Algorithms

In order to solve the nonlinear system of algebraic equations we developed the following two algorithms.

Algorithm 1. This algorithm is based on the *predictor-corrector* scheme and consists in the following steps, (see also [15,16] for the case of pricing American put options).

Step 1. Predictor. Let the solution and the free boundary position on the time level τ_j be known. Instead of (12) we use another approximation of (8) by introducing an artificial spatial node ξ_{-1} :

$$(1 + q(T - \tau_{j+1})) z^{j+1} = 1 + r(T - \tau_{j+1}) + \frac{\sigma^2}{2}(T - \tau_{j+1}) \frac{y_1^{j+1} - y_{-1}^{j+1}}{2h}. \quad (13)$$

An additional equation can be obtained from (5) by taking the limit $\xi \rightarrow 0$ and using the fact that $\partial_\tau \Pi(0, \tau) = 0$:

$$\alpha_0^{j+1} \frac{y_1^{j+1} - y_{-1}^{j+1}}{2h} - \frac{\sigma^2}{2} \frac{y_1^{j+1} - 2y_0^{j+1} + y_{-1}^{j+1}}{h^2} + \beta^{j+1} y_0^{j+1} = 0. \quad (14)$$

Using (13) we can express y_{-1}^{j+1} as:

$$y_{-1}^{j+1} = y_1^{j+1} - \left(qz^{j+1} - r + \frac{z^{j+1} - 1}{T - \tau_{j+1}} \right) \frac{4h}{\sigma^2}. \quad (15)$$

Inserting it into (14) we conclude the following equation for the value y_1^{j+1} :

$$y_1^{j+1} = \left(\frac{2\alpha_0^{j+1}h^2}{\sigma^4} + \frac{2h}{\sigma^2} \right) \left(qz^{j+1} - r + \frac{z^{j+1} - 1}{T - \tau_{j+1}} \right) - \frac{\beta^{j+1}h^2}{\sigma^2} - 1. \quad (16)$$

Instead of the implicit scheme (9) we make use of its explicit variant for $i = 1$ in order to derive

$$\frac{y_1^{j+1} - y_1^j}{k} + \alpha_1^{j+1} \frac{y_2^j - y_0^j}{2h} - \frac{\sigma^2}{2} \frac{y_2^j - 2y_1^j + y_0^j}{h^2} + \beta^{j+1} y_1^j = 0. \quad (17)$$

This way we obtain a nonlinear system (16), (17) for unknowns y_1^{j+1} and z^{j+1} . The system is indeed nonlinear as α_i^{j+1} depend on z^{j+1} . Now, by replacing $y_1^{j+1} \leftrightarrow \tilde{y}_1^{j+1}$ and $z^{j+1} \leftrightarrow \tilde{z}^{j+1}$ we construct the predictor value of \tilde{z}^{j+1} .

Step 2. Corrector. We again use Equation (9) in a slightly different form:

$$\frac{y_i^{j+1} - y_i^j}{k} + \hat{\alpha}_i^{j+1} \frac{y_{i+1}^{j+1} - y_{i-1}^{j+1}}{2h} - \frac{\sigma^2}{2} \frac{y_{i+1}^{j+1} - 2y_i^{j+1} + y_{i-1}^{j+1}}{h^2} + \beta^{j+1} y_i^{j+1} = 0, \quad (18)$$

where approximation $\hat{\alpha}_i^{j+1}$ takes into account the already constructed predictor value \tilde{z}^{j+1} , i.e.

$$\hat{\alpha}_i^{j+1} = \frac{\tilde{z}^{j+1} - z^j}{k\tilde{z}^{j+1}} + r - q - \frac{\sigma^2}{2} - \frac{\tilde{z}^{j+1} \exp(-\xi_i) - 1}{T - \tau_{j+1}}. \quad (19)$$

Next we use the corrected solution y_i^{j+1} and Equation (12) in order to obtain the corrected value for the free boundary position z^{j+1} on the next time layer.

Algorithm 2. We now describe an algorithm based on the *Newton method*. A variant of this method was applied for an American Call option problem in [5].

Step 1. We eliminate the known boundary values $y_0^{j+1} = -1$ and $y_N^{j+1} = 0$ from (9). Taking into account (12) we obtain a nonlinear system for N unknowns: y_i^{j+1} , $i = 1, 2, \dots, N-1$ and z^{j+1} . We denote by \mathbf{Y}^l the vector of these N unknowns at the l -th iteration.

Step 2. We have to solve the equation $\mathbf{F}^l = 0$ with $\mathbf{F}^l = \begin{pmatrix} \mathbf{F}_1^l & \mathbf{F}_2^l \end{pmatrix}^T$ where \mathbf{F}_i^l , $i = 1, 2$, correspond to Equations (9) and (12), respectively. To this end, we apply the Newton method in the following form: $\mathbf{J}^l (\mathbf{Y}^{l+1} - \mathbf{Y}^l) = -\mathbf{F}^l$, with the Jacobi matrix defined by: $\mathbf{J}^l = (\mathbf{J}_{ij}^l)_{i,j=1,2}$ where

$$\mathbf{J}_{11}^l = \begin{pmatrix} c_1^{j+1} & b_1^{j+1} & & & \\ a_2^{j+1} & c_2^{j+1} & b_2^{j+1} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N-2}^{j+1} & c_{N-2}^{j+1} & b_{N-2}^{j+1} \\ & & & a_{N-1}^{j+1} & c_{N-1}^{j+1} \end{pmatrix}, \quad \mathbf{J}_{12}^l = \begin{pmatrix} \frac{\partial a_1^{j+1}}{\partial z^{j+1}}(-1) + \frac{\partial b_1^{j+1}}{\partial z^{j+1}} y_2^{j+1} \\ \frac{\partial a_2^{j+1}}{\partial z^{j+1}} y_1^{j+1} + \frac{\partial b_2^{j+1}}{\partial z^{j+1}} y_3^{j+1} \\ \vdots \\ \frac{\partial a_{N-2}^{j+1}}{\partial z^{j+1}} y_{N-3}^{j+1} + \frac{\partial b_{N-2}^{j+1}}{\partial z^{j+1}} y_{N-1}^{j+1} \\ \frac{\partial a_{N-1}^{j+1}}{\partial z^{j+1}} y_{N-2}^{j+1} \end{pmatrix}$$

$\mathbf{J}_{21}^l = \left(\frac{-\sigma^2}{Dh}, \frac{\sigma^2}{4Dh}, 0, \dots, 0 \right)$ where $D = q + 1/(T - \tau^{j+1})$ and $\mathbf{J}_{22}^l = 1$. Similarly $\mathbf{Y}^l = \begin{pmatrix} \mathbf{Y}_1^l & \mathbf{Y}_2^l \end{pmatrix}^T$, $\mathbf{Y}_1^l = (y_1^{j+1}, \dots, y_{N-1}^{j+1})$, $\mathbf{Y}_2^l = z^{j+1}$. As for the elements of the matrix \mathbf{J}_{11}^l we have:

$$\begin{aligned} a_i^{j+1} &= -\frac{1}{2h} \left(\frac{z^{j+1} - z^j}{kz^{j+1}} + r - q - \frac{\sigma^2}{2} \right) - \frac{\sigma^2}{2h^2} + d_i^{j+1}, \\ c_i^{j+1} &= \frac{1}{k} + \frac{\sigma^2}{h^2} + r + \frac{1}{T - \tau_{j+1}}, \\ b_i^{j+1} &= \frac{1}{2h} \left(\frac{z^{j+1} - z^j}{kz^{j+1}} + r - q - \frac{\sigma^2}{2} \right) - \frac{\sigma^2}{2h^2} - d_i^{j+1}, \end{aligned}$$

and $d_i^{j+1} = 1/(2h)(z^{j+1} \exp(-\xi_i) - 1)/(T - \tau_{j+1})$. The iteration process is repeated until the condition $\| \mathbf{Y}^{l+1} - \mathbf{Y}^l \| < tol$ is fulfilled.

Step 3. The solution on the $(j+1)$ -th time layer is considered as an initial iteration for the next time layer. For solving $\mathbf{J} (\mathbf{Y}^{l+1} - \mathbf{Y}^l) = -\mathbf{F}^l$ we perform the following stages. First, we solve the linear system of equations $\mathbf{J}_{11}^{l+1} \mathbf{Y}_1^l = -\mathbf{F}_1^l + \mathbf{J}_{11}^l \mathbf{Y}_1^l - \mathbf{J}_{12}^{l+1} \mathbf{Y}_2^l + \mathbf{J}_{12}^l \mathbf{Y}_2^l$. Since the matrix \mathbf{J}_{11}^l is tridiagonal we can apply the Thomas algorithm to find \mathbf{Y}_1^{l+1} . Next, we solve $\mathbf{J}_{12}^{l+1} \mathbf{Y}_1^l + \mathbf{J}_{22}^{l+1} \mathbf{Y}_2^l = -\mathbf{F}_2^l$.

Remark 1. In both algorithms we choose the last time step $k - \varepsilon$ with $\varepsilon = 10^{-7}$, i.e. $\tau_M = T - \varepsilon$. To overcome possible numerical instabilities of these methods for $\tau \rightarrow T$ (i.e. $t \rightarrow 0$) we use the so called upwind and downwind approximations of the term $\frac{z^{j+1} \exp(-\xi_i) - 1}{T - \tau_{j+1}} \frac{\partial \Pi}{\partial \xi}$ depending of the sign of the term $z^{j+1} \exp(-\xi_i) - 1$.

5 Numerical Experiments

In this section we consider problem (1) with parameter values $r = 0.06$, $q = 0.04$, $\sigma = 0.2$ and $T = 50$, taken from examples presented in [1]. Since there exists no analytical solution to the proposed free boundary problem, we use the mesh refinement analysis with doubling the mesh size h . In Tab. 1 we present the position of the free boundary position $\rho(\tau)$ at different times τ constructed by the Newton method. We also present the difference between two consecutive values and the convergence ratio are presented. The results show nearly first order of accuracy for the free boundary and the CR increases with increasing τ (see Tab. 1). In Fig. 1a) a 3D plot of the portfolio function Π for $T = 50$, $N = 200$, $M = 500$ is presented. In Fig. 1b) the profiles of the function $\Pi(\xi, \tau)$ for $\tau = 0, 0.1, 10, 25, 50$ obtained by the Newton method are depicted.

In Fig. 2a) we show a comparison of the free boundary position $\rho(\tau)$ computed by our two algorithms (Predictor-corrector and Newton's based method) and by numerical methods from [1] (Bokes) and [2] (Kwok). It turns out that the Newton's based method gives nearly the same results as those of [1,2]. On the other hand, predictor-corrector methods slightly underestimates the free boundary position $\rho(\tau)$. In Fig. 2b) we show the free boundary position $x_f(t) = 1/\rho(T - t)$ for the original model variables $x = A/S$ and t . The continuation region and exercise region are also indicated.

6 Conclusions

In this paper we have analyzed numerical algorithms for solving the free boundary value problem for American style of floating strike Asian options. To solve corresponding degenerate parabolic problem we have applied Landau's front fixing transformation method. We proposed two numerical algorithms based on the predictor-corrector scheme and the Newton's method. The predictor-corrector

Table 1. Mesh-refinement analysis and the convergence ratio (CR) of the Newton method.

N	$ \rho(\tau = 10) $	difference	CR	$ \rho(\tau = 20) $	difference	CR	$ \rho(\tau = 40) $	difference	CR
50	1.949988	-	-	1.991675	-	-	1.796663	-	-
100	1.955552	5.5640e-3	-	1.995525	3.8502e-3	-	1.803276	6.6133e-3	-
200	1.958037	2.4850e-3	1.16	1.996945	1.4194e-3	1.44	1.805149	1.8729e-3	1.82
400	1.959199	1.1617e-3	1.10	1.997515	5.7099e-4	1.31	1.805667	5.1799e-4	1.85
800	1.959758	5.5965e-4	1.05	1.997765	2.4919e-4	1.20	1.805813	1.4621e-4	1.82

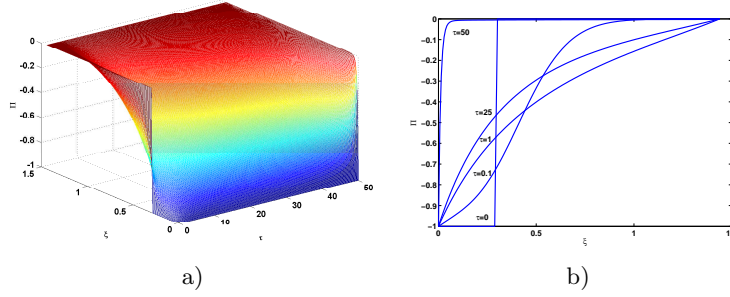


Fig. 1. (a) A 3D plot of the portfolio function Π for $T = 50$, $N = 200$, $M = 500$; (b) Profiles of the function $\Pi(\xi, \tau)$ for $\tau = 0$, $\tau = 0.1$, $\tau = 10$, $\tau = 25$, $\tau = 50$.

scheme is computationally faster when compared to the Newton method. It yields a good approximation close to expiry. However, its accuracy is decreased for times close to the initial time. The second algorithm based on Newton's method yields better approximation results over the whole time interval. Although all finite difference approximations are of second order, due to discontinuity of the initial datum and nonlinear behavior of the coefficients in all discrete equations, the results show nearly the first order rate of convergence.

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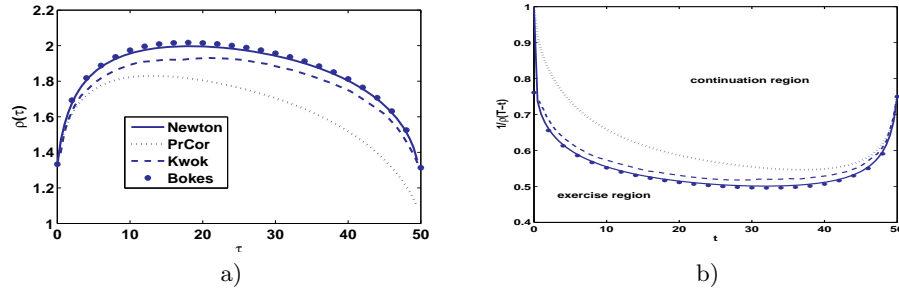


Fig. 2. a) Comparison of the free boundary $\rho(\tau)$ for various methods; b) the free boundary position $x_f(t) = 1/\rho(T - t)$ splitting the continuation and exercise region of American style of Asian call option.

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